

EXAMPLES OF FINITE DIMENSIONAL ALGEBRAS WHICH DO NOT SATISFY THE DERIVED JORDAN–HÖLDER PROPERTY

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ABSTRACT. We construct a matrix algebra $\Lambda(A, B)$ from two given finite dimensional elementary algebras A and B and give some sufficient conditions on A and B under which the derived Jordan–Hölder property (DJHP) fails for $\Lambda(A, B)$. This provides finite dimensional algebras of finite global dimension which do not satisfy DJHP.

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1. INTRODUCTION

Recollements of triangulated categories, defined in [5], provide a tool of decomposing triangulated categories into smaller ones. They are analogues of short exact sequences of groups or modules. In this paper we restrict our attention to recollements whose terms are all derived categories of algebras. Derived simple algebras, analogous to simple groups or simple modules, are those algebras whose derived categories cannot be decomposed further via recollements. For example, local algebras are derived simple [2]. In analogy to composition series, a stratification of a derived category is a sequence of recollements which iteratively decomposes the given category into derived simple ones. It is natural to ask for a derived version of Jordan–Hölder property (DJHP for short), that is, the existence and uniqueness of finite stratifications of derived categories. It was shown that hereditary Artin algebras [2], finite dimensional piecewise hereditary algebras [3], commutative Noetherian rings [4] and derived discrete algebras [18] satisfy DJHP.

DJHP fails for general algebras by results of Chen and Xi [8, 9]. The first finite dimensional example was proposed also by Chen and Xi. Another finite dimensional example was given in [4, Example 7.6]. The aim of this paper is to generalise this example to obtain more finite dimensional algebras which do not satisfy DJHP. More precisely, given two finite dimensional elementary algebras A and B , we construct a 2×2 -matrix algebra $\Lambda(A, B)$. Our main result is:

Theorem (3.8). *DJHP fails for $\Lambda(A, B)$ in the following two cases:*

- (1) *both A and B are derived simple, but neither is simple;*
- (2) *one of A and B is derived simple but nonsimple and the other one is of infinite global dimension.*

As consequences of this result, we have

- (a) There are finite dimensional algebras of finite global dimension which do not satisfy DJHP. Indeed, if A and B are finite dimensional derived simple elementary algebras of

positive finite global dimension, *e.g.* Fibonacci algebras [13, 17] and their generalisations [16], then $\Lambda(A, B)$ has finite global dimension and does not satisfy DJHP (Corollary 3.12).

- (b) Fix a local nonsimple finite dimensional elementary algebra A . Then for any finite dimensional elementary algebra B applying the construction $\Lambda(?) := \Lambda(A, ?)$ twice yields a finite dimensional algebra $\Lambda^2(B)$ which does not satisfy DJHP (Corollary 3.13).

During the preparation of this paper, Martin Kalck informed us that he found a family of finite dimensional algebras of global dimension 2 (hence quasi-hereditary) for which DJHP fails (see [14, Proposition 3.4]); Changchang Xi informed us that [10, Theorem 1.1] can be used to construct finite dimensional algebras which do not satisfy DJHP. For two finite dimensional elementary algebras with the same number of isomorphism classes of simple modules, Chen and Xi constructed in the end of [10, Section 5] an upper triangular 2×2 -matrix algebra, for which results similar to Theorem 3.8, (a) and (b) can be obtained.

The paper is organised as follows: In Section 2 we collect necessary definitions about recollements, stratifications and derived Jordan–Hölder property for later use. Section 3 contains our main result. We give the construction of the matrix algebra $\Lambda(A, B)$ and study its derived Jordan–Hölder property using stratifying ideals. In Section 4 we provide the underlying quiver and relations of $\Lambda(A, B)$.

Throughout the paper k is a field, all algebras are finite dimensional algebras over k , and all tensor products are over k unless otherwise stated. For an algebra A , we denote by $\mathcal{D}(A)$ the unbounded derived category of right A -modules. For a quiver Q , we denote by Q_0 and Q_1 the sets of vertices and the set of arrows respectively. For an arrow $\alpha \in Q_1$, we denote by $s(\alpha)$ and $t(\alpha)$ the source and target of α respectively.

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2. PRELIMINARIES

In this section we recall the definition of recollements, stratifying ideals and stratifying recollements, stratifications, derived simplicity and derived Jordan–Hölder property of algebras.

2.1. Recollements. Let A, B, C be algebras. A *recollement* [5] of $\mathcal{D}(A)$ by $\mathcal{D}(B)$ and $\mathcal{D}(C)$ is a diagram

$$\begin{array}{ccccc} \xleftarrow{i^*} & & \xleftarrow{j_!} & & \\ \mathcal{D}(B) & \xrightarrow{i_* = i_!} & \mathcal{D}(A) & \xrightarrow{j^! = j^*} & \mathcal{D}(C) \\ \xleftarrow{i^!} & & \xleftarrow{j_*} & & \end{array}$$

of k -linear triangle functors such that

- (1) (i^*, i_*) , $(i_!, i^!)$, $(j_!, j^!)$, (j^*, j_*) are adjoint pairs;
- (2) i_* , j_* , $j_!$ are full embeddings;
- (3) $i^! \circ j_* = 0$ (and thus also $j^! \circ i_! = 0$ and $i^* \circ j_! = 0$);

(4) for each $M \in \mathcal{D}(A)$ there are triangles

$$\begin{aligned} i_! i^!(M) &\longrightarrow M \longrightarrow j_* j^*(M) \longrightarrow i_! i^!(M)[1] \\ j_! j^!(M) &\longrightarrow M \longrightarrow i_* i^*(M) \longrightarrow j_! j^!(M)[1] \end{aligned}$$

where the first two maps in each triangle are given by adjunctions.

We say the recollement is *trivial* if one of the algebras B and C is trivial, or equivalently, one of the full embeddings i_* , $j_!$ and j_* is a triangle equivalence.

For an algebra A , we denote by $\text{rank}(A)$ the number of nonisomorphic simple A -modules and call it the *rank* of A .

Proposition 2.1. *Let A, B, C be algebras and assume that there is a recollement of $\mathcal{D}(A)$ by $\mathcal{D}(B)$ and $\mathcal{D}(C)$. Then*

- (i) ([4, Proposition 2.14]) *A is of finite global dimension if and only if so are B and C ;*
- (ii) ([4, Proposition 6.5]) $\text{rank}(A) = \text{rank}(B) + \text{rank}(C)$.

2.2. Stratifying ideals and stratifying recollements. Let A be an algebra and $e \in A$ an idempotent. Following [6, Definition 2.1.1] we call the idempotent ideal AeA a *stratifying ideal* of A if the natural morphism $Ae \overset{L}{\otimes}_{eAe} eA \rightarrow AeA$ is an isomorphism in $\mathcal{D}(A)$. If AeA is stratifying, then there exists a recollement of $\mathcal{D}(A)$ by $\mathcal{D}(A/AeA)$ and $\mathcal{D}(eAe)$ (see for example [1, Example 4.5])

$$\begin{array}{ccccc} \mathcal{D}(A/AeA) & \overset{i^*}{\longleftarrow} & \mathcal{D}(A) & \overset{j_!}{\longleftarrow} & \mathcal{D}(eAe) \\ & \xrightarrow{i_* = i_!} & & \xrightarrow{j^* = j^!} & \\ & \underset{i^!}{\longleftarrow} & & \underset{j_*}{\longleftarrow} & \end{array}$$

where the six functors are derived functors given by

$$\begin{aligned} i^* &= - \overset{L}{\otimes}_A A/AeA, \quad i^! = \mathbf{R}\text{Hom}_A(A/AeA, -), \\ i_* &= \mathbf{R}\text{Hom}_{A/AeA}(A/AeA, -) = - \overset{L}{\otimes}_{A/AeA} A/AeA = i_!, \\ j_! &= - \overset{L}{\otimes}_{eAe} eA, \quad j_* = \mathbf{R}\text{Hom}_{eAe}(Ae, -), \\ j^! &= \mathbf{R}\text{Hom}_A(eA, -) = - \overset{L}{\otimes}_A Ae = j^*. \end{aligned}$$

We refer to it as the *stratifying recollement* of $\mathcal{D}(A)$ induced by e . The following lemma is a left-module version of [4, Lemma 2.12].

Lemma 2.2. *Let A be an algebra and $e \in A$ an idempotent. Assume that A/AeA , as a left A -module, admits a projective resolution with components in $\mathbf{add}(Ae)$ except in degree 0. Then AeA is a stratifying ideal of A .*

2.3. Derived simplicity of algebras. An algebra A is said to be *derived simple* if there is no nontrivial recollement of $\mathcal{D}(A)$ by derived categories of algebras. Known examples include local algebras [2, Proposition 4.11], Fibonacci algebras [13, 17] and their generalisations [16], blocks of group algebras of finite groups and indecomposable symmetric algebras of finite representation type [15], and indecomposable self-injective algebras of rank 2 with nonpositive Cartan determinant [19, Remark 5].

2.4. Stratifications and the derived Jordan–Hölder property. Roughly speaking, a *stratification* is a way of breaking up a given derived category into simple pieces using recollements. More rigorously, for an algebra A a stratification of $\mathcal{D}(A)$ is a full rooted binary tree whose root is the given derived category $\mathcal{D}(A)$, whose nodes are derived categories of algebras and whose leaves are derived categories of derived simple algebras such that a node together with its two child nodes form a nontrivial recollement unless it is a leaf. The leaves are called the *simple factors* of the stratification. By abuse of language, we will also call the corresponding derived simple algebras the simple factors of the stratification. If we allow trivial recollements in the above definition, we call the binary tree a *pseudostratification*. Given a pseudostratification, we obtain a stratification by deleting the subtrees with trivial nodes and contracting the edges which are equivalences.

Lemma 2.3. *Let A be an algebra with a decomposition $A = A_1 \times \cdots \times A_s$ into the direct product of algebras A_1, \dots, A_s . Then any stratification of $\mathcal{D}(A)$ is the direct sum of pseudostratifications of $\mathcal{D}(A_1), \dots, \mathcal{D}(A_s)$.*

Proof. Let B and C be algebras such that there is a recollement of $\mathcal{D}(A)$ by $\mathcal{D}(B)$ and $\mathcal{D}(C)$. It follows from [4, Lemma 2.3] that there are algebras $B_1, \dots, B_s, C_1, \dots, C_s$ (some of which may be trivial) such that $B = B_1 \times \cdots \times B_s$ and $C = C_1 \times \cdots \times C_s$ and there are recollements of $\mathcal{D}(A_i)$ by $\mathcal{D}(B_i)$ and $\mathcal{D}(C_i)$, $i = 1, \dots, s$, which sum up to the given recollement. The desired result follows by induction. \checkmark

Analogous to the classical Jordan–Hölder property of finite groups, the derived Jordan–Hölder property of an algebra is about the existence and uniqueness of finite stratifications. Let A be an algebra. According to [4, Corollary 7.4], any stratification of the derived category $\mathcal{D}(A)$ is finite. If the simple factors in any stratification of $\mathcal{D}(A)$ are unique up to ordering and up to derived equivalence, we say that the algebra A satisfies the *derived Jordan–Hölder property* (DJHP for short). As examples, piecewise hereditary algebras [2, 3] and derived discrete algebras [18] satisfy DJHP.

Proposition 2.4. *Assume that k is algebraically closed. If A is of rank 2 and of finite global dimension, then A satisfies DJHP.*

Proof. Assume that A is not derived simple. Then there are algebras B and C such that there is a nontrivial recollement of $\mathcal{D}(A)$ by $\mathcal{D}(B)$ and $\mathcal{D}(C)$. By Proposition 2.1, both B and C are of rank 1 and of finite global dimension, so they are derived equivalent to k . Therefore, any stratification of $\mathcal{D}(A)$ is a recollement and has simple factors two copies of k . \checkmark

According to [16, Theorem 3.7], an algebra over an algebraically closed field which is of rank 2 and of finite global dimension is either derived simple or derived equivalent to a quasi-hereditary algebra.

3. MAIN RESULT

In this section we construct a matrix algebra $\Lambda = \Lambda(A, B)$ from two elementary algebras A and B . We show that it has two natural stratifying ideals. By analysing the factors in the induced stratifying recollements, we give two cases when DJHP fails for Λ (see Theorem 3.8).

An algebra is said to be *elementary* if all its simple modules are 1-dimensional, equivalently, if it is isomorphic to the quotient of the path algebra of a finite quiver by an admissible ideal. For an elementary algebra A let $\text{rad}(A)$ be the radical of the algebra A and $\bar{A} = A/\text{rad}(A)$ the semisimple quotient, which is the direct product of finite copies of k . We fix a retraction $\bar{A} \hookrightarrow A$ such that \bar{A} becomes a subalgebra of A (if A is given by a quiver with relations, we take \bar{A} to be the subspace spanned by all trivial paths). Then A is isomorphic to $\bar{A} \oplus \text{rad}(A)$ as a k -vector space. For $a \in A$, we will write $a = a_1 + a_2$ with $a_1 \in \bar{A}$ and $a_2 \in \text{rad}(A)$.

3.1. The trivially twisted tensor product. Let A and B be two elementary algebras. Define a k -linear map $\tau : B \otimes A \rightarrow A \otimes B$ as the composition

$$B \otimes A \twoheadrightarrow (B \otimes A)/(\text{rad}(B) \otimes \text{rad}(A)) \xrightarrow{\text{flip}} (A \otimes B)/(\text{rad}(A) \otimes \text{rad}(B)) \hookrightarrow A \otimes B$$

where the flip is defined by switching the two components in the tensor products, $(A \otimes B)/(\text{rad}(A) \otimes \text{rad}(B))$ is identified with $\bar{A} \otimes \bar{B} \oplus \bar{A} \otimes \text{rad}(B) \oplus \text{rad}(A) \otimes \bar{B}$ and naturally embedded into $A \otimes B$. Thus for $a \in A$ and $b \in B$ we have

$$\tau(b \otimes a) = a \otimes b_1 + a_1 \otimes b_2 = a_1 \otimes b_1 + a_2 \otimes b_1 + a_1 \otimes b_2 = a \otimes b - a_2 \otimes b_2.$$

In particular, $\tau(b \otimes a) = 0$ if $a \in \text{rad}(A)$ and $b \in \text{rad}(B)$.

Denote by μ_A and μ_B the multiplications of the algebras A and B , respectively. Define a k -linear map μ_τ on the tensor product $A \otimes B$:

$$\mu_\tau : (A \otimes B) \otimes (A \otimes B) \xrightarrow{\text{id}_A \otimes \tau \otimes \text{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B.$$

More precisely, for $a, a' \in A$ and $b, b' \in B$, we have

$$\begin{aligned} \mu_\tau((a \otimes b) \otimes (a' \otimes b')) &= \mu_A \otimes \mu_B(a \otimes (a' \otimes b_1 + a'_1 \otimes b_2) \otimes b') \\ &= aa' \otimes b_1 b' + aa'_1 \otimes b_2 b' \\ &= aa' \otimes bb' - aa'_2 \otimes b_2 b'. \end{aligned}$$

Proposition 3.1. *The vector space $A \otimes B$ with multiplication μ_τ is an algebra.*

Proof. By [7, Proposition/Definition 2.3 and Remark 2.4 (1)], it is enough to prove

$$\begin{aligned} \tau \circ (\text{id}_B \otimes \mu_A) &= (\mu_A \otimes \text{id}_B) \circ (\text{id}_A \otimes \tau) \circ (\tau \otimes \text{id}_A), \\ \tau \circ (\mu_B \otimes \text{id}_A) &= (\text{id}_A \otimes \mu_B) \circ (\tau \otimes \text{id}_B) \circ (\text{id}_B \otimes \tau). \end{aligned}$$

We only proof the first equality and the second is similar. Let $b \in B$ and $a, a' \in A$. Then

$$\begin{aligned} \tau \circ (\text{id}_B \otimes \mu_A)(b \otimes a \otimes a') &= \tau(b \otimes aa') \\ &= \tau((b_1 + b_2) \otimes (a_1 a'_1 + (a_1 a'_2 + a_2 a'_1 + a_2 a'_2))) \\ &= aa' \otimes b_1 + a_1 a'_1 \otimes b_2. \end{aligned}$$

On the other hand

$$\begin{aligned}
(\mu_A \otimes \text{id}_B) \circ (\text{id}_A \otimes \tau) \circ (\tau \otimes \text{id}_A)(b \otimes a \otimes a') &= (\mu_A \otimes \text{id}_B) \circ (\text{id}_A \otimes \tau)((a \otimes b_1 + a_1 \otimes b_2) \otimes a') \\
&= (\mu_A \otimes \text{id}_B)(a \otimes a' \otimes b_1 + a_1 \otimes a' \otimes b_2) \\
&= aa' \otimes b_1 + a_1 a'_1 \otimes b_2.
\end{aligned}$$

✓

Following [10] we call the algebra in Proposition 3.1 the *trivially twisted tensor product* of A and B (over k) and denote it by $A \otimes_0 B$. If A or B is semisimple, then $A \otimes_0 B$ is the usual tensor product $A \otimes B$ of algebras. Moreover, $\bar{A} \otimes B$ and $A \otimes \bar{B}$ are subalgebras of $A \otimes_0 B$.

Lemma 3.2. (i) $A \otimes \text{rad}(B)$ and $\text{rad}(A) \otimes B$ are two-sided ideals of $A \otimes_0 B$ with quotients $A \otimes \bar{B}$ and $\bar{A} \otimes B$, respectively.

(ii) The restriction of μ_τ to $(A \otimes \text{rad}(B)) \otimes (\text{rad}(A) \otimes B)$ vanishes.

Proof. (i) We check that $A \otimes \text{rad}(B)$ is closed when multiplying with an element in $A \otimes_0 B$. The proof for $\text{rad}(A) \otimes B$ is similar. For $a, a' \in A$ and $b = b_2 \in \text{rad}(B), b' \in B$ we have

$$\begin{aligned}
\mu_\tau((a \otimes b_2) \otimes (a' \otimes b')) &= \mu_A \otimes \mu_B(a \otimes a'_1 \otimes b_2 \otimes b') \\
&= aa'_1 \otimes b_2 b' \in A \otimes \text{rad}(B),
\end{aligned}$$

and

$$\begin{aligned}
\mu_\tau((a' \otimes b') \otimes (a \otimes b_2)) &= \mu_A \otimes \mu_B(a' \otimes (a \otimes b'_1 + a_1 \otimes b'_2) \otimes b_2) \\
&= a'a \otimes b'_1 b_2 + a'a_1 \otimes b'_2 b_2 \in A \otimes \text{rad}(B).
\end{aligned}$$

(ii) This follows from the fact that the restriction of τ to $\text{rad}(B) \otimes \text{rad}(A)$ vanishes. ✓

3.2. The matrix algebra: construction and properties. Let A and B be two elementary algebras. We have the following data:

- (1) an $(\bar{A} \otimes B)$ -($A \otimes_0 B$)-bimodule $\bar{A} \otimes B$: the left $(\bar{A} \otimes B)$ -module structure is the free module of rank 1; the right $(A \otimes_0 B)$ -module structure is induced from the quotient map $A \otimes_0 B \twoheadrightarrow \bar{A} \otimes B = (A \otimes_0 B)/(\text{rad}(A) \otimes B)$ in Lemma 3.2 (i); explicitly, we have

$$(a' \otimes b')(a \otimes b)(a'' \otimes b'') = a'aa'' \otimes b'bb''$$

for $a \otimes b, a' \otimes b' \in \bar{A} \otimes B$ and $a'' \otimes b'' \in A \otimes_0 B$;

- (2) an $(A \otimes_0 B)$ -($\bar{A} \otimes B$)-bimodule $A \otimes \text{rad}(B)$: $A \otimes \text{rad}(B)$ as a two-sided ideal of $A \otimes_0 B$ is naturally an $(A \otimes_0 B)$ -($A \otimes_0 B$)-bimodule; by Lemma 3.2 (ii), $\text{rad}(A) \otimes B$ annihilates $A \otimes \text{rad}(B)$ from the right hand side, hence the right $(A \otimes_0 B)$ -module structure on $A \otimes \text{rad}(B)$ factors through $\bar{A} \otimes B$ via the quotient map $A \otimes_0 B \twoheadrightarrow \bar{A} \otimes B = (A \otimes_0 B)/(\text{rad}(A) \otimes B)$ in Lemma 3.2 (i); the right $\bar{A} \otimes B$ -module structure on $A \otimes \text{rad}(B)$ can also be obtained via the embedding $\bar{A} \otimes B \hookrightarrow A \otimes_0 B$ because the composition of this embedding with the quotient map $A \otimes_0 B \twoheadrightarrow \bar{A} \otimes B$ is the identity of $\bar{A} \otimes B$; explicitly, we have

$$(a' \otimes b')(a \otimes b)(a'' \otimes b'') = a'aa'' \otimes b'_1 bb'' + a'a_1 a'' \otimes b'_2 bb''$$

for $a \otimes b \in A \otimes \text{rad}(B)$, $a' \otimes b' \in A \otimes_0 B$ and $a'' \otimes b'' \in \bar{A} \otimes B$;

(3) an $(A \otimes_0 B)$ -($A \otimes_0 B$)-bimodule homomorphism

$$\varphi : (A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes B) \rightarrow A \otimes_0 B, (a \otimes b) \otimes (a' \otimes b') \mapsto aa' \otimes bb'$$

for $a \otimes b \in A \otimes \text{rad}(B)$ and $a' \otimes b' \in \bar{A} \otimes B$; it is the composition of the canonical isomorphism $(A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes B) \xrightarrow{\cong} A \otimes \text{rad}(B)$ and the embedding $A \otimes \text{rad}(B) \hookrightarrow A \otimes_0 B$;

(4) an $(\bar{A} \otimes B)$ -($\bar{A} \otimes B$)-bimodule homomorphism

$$\psi : (\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes \text{rad}(B)) \rightarrow \bar{A} \otimes B, (a \otimes b) \otimes (a' \otimes b') \mapsto aa'_1 \otimes bb'$$

for $a \otimes b \in \bar{A} \otimes B$ and $a' \otimes b' \in A \otimes \text{rad}(B)$; it is the composition of the embedding $(\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes \text{rad}(B)) \hookrightarrow (\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes_0 B)$ with the canonical isomorphism $(\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes_0 B) \xrightarrow{\cong} \bar{A} \otimes B$.

Moreover, the following two diagrams are commutative

$$\begin{array}{ccc} (\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes B) & \xrightarrow{\text{id}_{\bar{A} \otimes B} \otimes \varphi} & (\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes_0 B) \\ \downarrow \psi \otimes \text{id}_{\bar{A} \otimes B} & & \downarrow \cong \\ (\bar{A} \otimes B) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes B) & \xrightarrow{\cong} & \bar{A} \otimes B \\ \\ (A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes \text{rad}(B)) & \xrightarrow{\text{id}_{A \otimes \text{rad}(B)} \otimes \psi} & (A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes B) \\ \downarrow \varphi \otimes \text{id}_{A \otimes \text{rad}(B)} & & \downarrow \cong \\ (A \otimes_0 B) \otimes_{A \otimes_0 B} (A \otimes \text{rad}(B)) & \xrightarrow{\cong} & A \otimes \text{rad}(B). \end{array}$$

We check the commutativity of the first diagram, and the commutativity of all other diagrams in the section can be checked in a similar way. For $a \otimes b \in \bar{A} \otimes B$, $a' \otimes b' \in A \otimes \text{rad}(B)$ and $a'' \otimes b'' \in \bar{A} \otimes B$, we have

$$(a \otimes b) \varphi((a' \otimes b') \otimes (a'' \otimes b'')) = (a \otimes b)(a'a'' \otimes b'b'') = aa'_1 a'' \otimes bb'b'',$$

$$\psi((a \otimes b) \otimes (a' \otimes b'))(a'' \otimes b'') = (aa'_1 \otimes bb')(a'' \otimes b'') = aa'_1 a'' \otimes bb'b''.$$

Therefore we obtain an algebra

$$\Lambda(A, B) := \begin{pmatrix} A \otimes_0 B & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes B \end{pmatrix},$$

whose addition is componentwise and whose multiplication is given by matrix multiplication together with the homomorphisms φ and ψ . See for example [12] for the study on matrix rings. In Section 4 we will describe the quiver with relations of $\Lambda(A, B)$ in terms of those of A and B .

Put $\Lambda = \Lambda(A, B)$. By abuse of notation we write 1 for the unit elements in all of the algebras A , B and Λ . Consider the two natural idempotents in Λ

$$e_1 = \begin{pmatrix} 1 \otimes 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix}.$$

Then in Λ it holds that $1 = e_1 + e_2$ and $e_1 e_2 = 0 = e_2 e_1$. Since A is elementary, the rank of A equals $\dim \bar{A}$. The rank of a general algebra is smaller than or equal to the cardinality of a complete set of primitive orthogonal idempotents of A ; the equality holds if it is elementary.

Lemma 3.3. *The following hold for Λ :*

- (i) $e_1 \Lambda e_1 = A \otimes_0 B$ and $\Lambda / \Lambda e_1 \Lambda \cong \bar{A} \otimes \bar{B}$;
- (ii) $e_2 \Lambda e_2 = \bar{A} \otimes B$ and $\Lambda / \Lambda e_2 \Lambda \cong A \otimes \bar{B}$;
- (iii) Λ is elementary with $\text{rank}(\Lambda) = 2\text{rank}(A)\text{rank}(B)$.

Proof. (i) It is clear that $e_1 \Lambda e_1 = A \otimes_0 B$. The ideal generated by e_1 is

$$\begin{aligned} \Lambda e_1 \Lambda &= \begin{pmatrix} A \otimes_0 B & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes B \end{pmatrix} \begin{pmatrix} 1 \otimes 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A \otimes_0 B & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes B \end{pmatrix} \\ &= \begin{pmatrix} A \otimes_0 B & 0 \\ \bar{A} \otimes B & 0 \end{pmatrix} \begin{pmatrix} A \otimes_0 B & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes B \end{pmatrix} \\ &= \begin{pmatrix} A \otimes_0 B & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes \text{rad}(B) \end{pmatrix}. \end{aligned}$$

Hence $\Lambda / \Lambda e_1 \Lambda \cong \bar{A} \otimes B / (\bar{A} \otimes \text{rad}(B)) \cong \bar{A} \otimes \bar{B}$.

(ii) Similar to (i).

(iii) Let $\{e_i \mid i \in I\}$ and $\{f_j \mid j \in J\}$ be complete sets of primitive orthogonal idempotents of \bar{A} and \bar{B} , respectively. Then $e_i \otimes f_j$ ($i \in I, j \in J$) is a primitive idempotent of $A \otimes_0 B$ because $(e_i \otimes f_j)(A \otimes_0 B)(e_i \otimes f_j) = e_i A e_i \otimes_0 f_j B f_j$ is local. It follows that

$$\left\{ \begin{pmatrix} e_i \otimes f_j & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & e_i \otimes f_j \end{pmatrix} \mid i \in I, j \in J \right\}$$

is a complete set of primitive orthogonal idempotents of Λ . It follows that $\text{rank}(\Lambda) \leq 2\text{rank}(A)\text{rank}(B)$.

On the other hand, the subspace

$$I = \begin{pmatrix} A \otimes \text{rad}(B) + \text{rad}(A) \otimes B & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes \text{rad}(B) \end{pmatrix}$$

is an ideal of Λ with semisimple quotient algebra

$$\Lambda / I = \begin{pmatrix} \bar{A} \otimes \bar{B} & 0 \\ 0 & \bar{A} \otimes \bar{B} \end{pmatrix}.$$

Therefore $\text{rank}(\Lambda) \geq 2\text{rank}(A)\text{rank}(B)$. Altogether we have $\text{rank}(\Lambda) = 2\text{rank}(A)\text{rank}(B)$ and Λ is elementary. Moreover, $I = \text{rad}(\Lambda)$ and $\bar{\Lambda} = \Lambda / I$. ✓

Sometimes we can recover B from Λ .

Lemma 3.4. *Assume that A is local nonsimple and let B' be another elementary algebra. If $\Lambda = \Lambda(A, B)$ and $\Lambda(A, B')$ are isomorphic as algebras, then B and B' are isomorphic as algebras.*

Proof. Put $\Lambda' = \Lambda(A, B')$ and suppose that $\phi : \Lambda \rightarrow \Lambda'$ is an algebra isomorphism. By Lemma 3.3 (iii), B and B' have the same rank. Let $\{f_1, \dots, f_m\}$ and $\{f'_1, \dots, f'_m\}$ be complete sets of primitive orthogonal idempotents of \bar{B} and \bar{B}' , respectively. Then

$$\left\{ \begin{pmatrix} 1 \otimes f_j & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes f_j \end{pmatrix} \mid j = 1, \dots, m \right\}$$

and

$$\left\{ \begin{pmatrix} 1 \otimes f'_j & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes f'_j \end{pmatrix} \mid j = 1, \dots, m \right\}$$

are complete sets of primitive orthogonal idempotents of Λ and Λ' , respectively. Moreover, for $1 \leq j \leq m$, the ideal of Λ generated by $1 - \begin{pmatrix} 1 \otimes f_j & 0 \\ 0 & 0 \end{pmatrix}$ is

$$\Lambda(1 - \begin{pmatrix} 1 \otimes f_j & 0 \\ 0 & 0 \end{pmatrix})\Lambda = \begin{pmatrix} A \otimes (\text{rad}(B) + B(1 - f_j)B) & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes B \end{pmatrix},$$

so

$$\Lambda/\Lambda(1 - \begin{pmatrix} 1 \otimes f_j & 0 \\ 0 & 0 \end{pmatrix})\Lambda \cong A.$$

Similarly, for $1 \leq j \leq m$, we have

$$\Lambda/\Lambda(1 - \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes f_j \end{pmatrix})\Lambda \cong k,$$

$$\Lambda'/\Lambda'(1 - \begin{pmatrix} 1 \otimes f'_j & 0 \\ 0 & 0 \end{pmatrix})\Lambda' \cong A,$$

$$\Lambda'/\Lambda'(1 - \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes f'_{j'} \end{pmatrix})\Lambda' \cong k.$$

Therefore, it follows from [11, Theorem 3.4.1] that there exist an invertible element $\lambda \in \Lambda'$ and a permutation σ on $\{1, \dots, m\}$ such that $\phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes f_j \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes f'_{\sigma(j)} \end{pmatrix} \lambda^{-1}$ for any $1 \leq$

$j \leq m$. So $\phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix} \lambda^{-1}$, and we obtain a chain of algebra isomorphisms

$$\begin{aligned} B &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix} \Lambda \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix} \\ &\cong \phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix} \Lambda' \phi \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix} \\ &\cong \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix} \Lambda' \begin{pmatrix} 0 & 0 \\ 0 & 1 \otimes 1 \end{pmatrix} \\ &= B'. \end{aligned}$$

✓

3.3. The matrix algebra: failure of DJHP. Let A and B be two elementary algebras. In Section 3.2 we defined a 2×2 -matrix algebra $\Lambda = \Lambda(A, B)$ and two idempotents e_1 and e_2 of Λ .

Proposition 3.5. *The idempotent ideals generated by e_1 and e_2 are both stratifying ideals of Λ . Consequently, $\mathcal{D}(\Lambda)$ has two stratifying recollements, induced by e_1 and e_2 and denoted by (R1) and (R2) respectively, such that*

- (R1) is a recollement of $\mathcal{D}(\Lambda)$ by $\mathcal{D}(\Lambda/\Lambda e_1 \Lambda)$ and $\mathcal{D}(e_1 \Lambda e_1)$, where $\Lambda/\Lambda e_1 \Lambda \cong \bar{A} \otimes \bar{B}$, which is the direct product of $\text{rank}(A)\text{rank}(B)$ copies of k , and $e_1 \Lambda e_1 = A \otimes_0 B$;
- (R2) is a recollement of $\mathcal{D}(\Lambda)$ by $\mathcal{D}(\Lambda/\Lambda e_2 \Lambda)$ and $\mathcal{D}(e_2 \Lambda e_2)$, where $\Lambda/\Lambda e_2 \Lambda \cong A \otimes \bar{B}$, which is the direct product of $\text{rank}(B)$ copies of A , and $e_2 \Lambda e_2 = \bar{A} \otimes B$, which is the direct product of $\text{rank}(A)$ copies of B .

We need some preparation for the proof of Proposition 3.5. Let M be a left B -module with B -action given by $\rho_M : B \otimes M \rightarrow M$. By [7, Sections 3.6, 3.7 and 3.8], the two equalities in the proof of Proposition 3.1 shows that the map

$$\begin{aligned} (A \otimes B) \otimes (A \otimes M) &\xrightarrow{\text{id}_A \otimes \tau \otimes \text{id}_M} A \otimes A \otimes B \otimes M \xrightarrow{\mu_A \otimes \rho_M} A \otimes M \\ (a \otimes b) \otimes (a' \otimes m) &\longmapsto aa' \otimes b_1 m + aa'_1 \otimes b_2 m \end{aligned}$$

defines a left $(A \otimes_0 B)$ -module structure on $A \otimes M$. We denote it by $A \otimes_0 M$. Note that the ideal $A \otimes \text{rad}(B)$ of $A \otimes_0 B$ as a left $(A \otimes_0 B)$ -module is $A \otimes_0 \text{rad}(B)$. For a left B -module homomorphism $f : M \rightarrow N$, the map

$$\text{id}_A \otimes f : A \otimes_0 M \longrightarrow A \otimes_0 N, \quad a \otimes m \mapsto a \otimes f(m)$$

is a left $A \otimes_0 B$ -module homomorphism.

Next we construct a left Λ -module from M . For knowledge on modules over matrix rings, we refer to [12]. Now we have a left $(A \otimes_0 B)$ -module $A \otimes_0 M$ and a left $(\bar{A} \otimes B)$ -module $\bar{A} \otimes M$. Moreover, we have

- a left $A \otimes_0 B$ -module homomorphism

$$\xi = \xi_M : (A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes M) \rightarrow A \otimes_0 M, \quad (a \otimes b) \otimes (a' \otimes m) \mapsto aa' \otimes bm,$$

for $a \otimes b \in A \otimes \text{rad}(B)$ and $a' \otimes b' \in \bar{A} \otimes M$,

- a left $\bar{A} \otimes B$ -module homomorphism

$$\eta = \eta_M : (\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes_0 M) \rightarrow \bar{A} \otimes M, \quad (a \otimes b) \otimes (a' \otimes m) \mapsto aa'_1 \otimes bm,$$

for $a \otimes b \in \bar{A} \otimes B$ and $a' \otimes m \in A \otimes_0 M$.

They satisfy the following commutative diagrams

$$\begin{array}{ccc}
(\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes M) & \xrightarrow{\text{id}_{\bar{A} \otimes B} \otimes \xi} & (\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes_0 M) \\
\downarrow \psi \otimes \text{id}_{\bar{A} \otimes M} & & \downarrow \eta \\
(\bar{A} \otimes B) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes M) & \xrightarrow{\cong} & \bar{A} \otimes M \\
\\
(A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes B) \otimes_{A \otimes_0 B} (A \otimes_0 M) & \xrightarrow{\text{id}_{A \otimes \text{rad}(B)} \otimes \eta} & (A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes M) \\
\downarrow \varphi \otimes \text{id}_{A \otimes_0 M} & & \downarrow \xi \\
(A \otimes_0 B) \otimes_{A \otimes_0 B} (A \otimes_0 M) & \xrightarrow{\cong} & A \otimes_0 M.
\end{array}$$

So ξ and η define a left Λ -module structure on

$$\begin{pmatrix} A \otimes_0 M \\ \bar{A} \otimes M \end{pmatrix}.$$

We denote this module by \widetilde{M} . Note that \widetilde{B} is naturally isomorphic to Λe_1 as a left Λ -module.

Let $f : M \rightarrow N$ be a left B -module homomorphism. Then there are commutative diagrams

$$\begin{array}{ccc}
(\bar{A} \otimes B)_{A \otimes_0 B} (A \otimes_0 M) & \xrightarrow{\eta_M} & \bar{A} \otimes M \\
\downarrow \text{id}_{\bar{A} \otimes B} \otimes (\text{id}_A \otimes f) & & \downarrow \text{id}_{\bar{A}} \otimes f \\
(\bar{A} \otimes B)_{A \otimes_0 B} (A \otimes_0 N) & \xrightarrow{\eta_N} & \bar{A} \otimes N \\
\\
(A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes M) & \xrightarrow{\xi_M} & A \otimes_0 M \\
\downarrow (\text{id}_{A \otimes \text{rad}(B)}) \otimes (\text{id}_{\bar{A}} \otimes f) & & \downarrow \text{id}_A \otimes f \\
(A \otimes \text{rad}(B)) \otimes_{\bar{A} \otimes B} (\bar{A} \otimes N) & \xrightarrow{\xi_N} & A \otimes_0 N.
\end{array}$$

It follows that

$$\tilde{f} := \begin{pmatrix} \text{id}_A \otimes f & \\ & \text{id}_{\bar{A}} \otimes f \end{pmatrix} : \widetilde{M} \longrightarrow \widetilde{N}$$

is a left Λ -module homomorphism. Therefore, for a complex

$$M^\bullet = \dots \rightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \rightarrow \dots$$

of left B -modules, the sequence

$$\widetilde{M}^\bullet := \dots \rightarrow \widetilde{M}^{i-1} \xrightarrow{\widetilde{d}^{i-1}} \widetilde{M}^i \xrightarrow{\widetilde{d}^i} \widetilde{M}^{i+1} \rightarrow \dots$$

is a complex of left Λ -modules. Moreover, M^\bullet is acyclic if and only if so is \widetilde{M}^\bullet because as a complex of vector spaces \widetilde{M}^\bullet is just $(A \oplus \bar{A}) \otimes M^\bullet$. Therefore we have

Lemma 3.6. *Let M be a left B -module and let P^\bullet be a free resolution of M over B . Then \widetilde{P}^\bullet is a projective resolution of \widetilde{M} over Λ with all components in $\text{add}(\Lambda e_1)$.*

Proof of Proposition 3.5. The second statement follows from the first statement together with Lemma 3.3. To prove the first statement, we construct a projective resolution of the left Λ -module $\Lambda/\Lambda e_i \Lambda$ ($i = 1, 2$) all of whose components but the one in degree 0 belong to $\text{add}(\Lambda e_i)$. The statement follows then by Lemma 2.2.

Multiplying $\begin{pmatrix} 0 & 0 \\ 1 \otimes 1 & 0 \end{pmatrix}$ on the right defines an injective Λ -module homomorphism

$$\Lambda e_2 \longrightarrow \Lambda e_1$$

with quotient $\Lambda/\Lambda e_2\Lambda$. So there is a short exact sequence

$$0 \rightarrow \Lambda e_2 \rightarrow \Lambda e_1 \rightarrow \Lambda/\Lambda e_2\Lambda \rightarrow 0,$$

which is a projective Λ -resolution of $\Lambda/\Lambda e_2\Lambda$ with the desired property.

Consider the projection from Λe_2 to $\Lambda/\Lambda e_1\Lambda$. Its kernel is naturally isomorphic to

$$\widetilde{\text{rad}(B)} = \begin{pmatrix} A \otimes_0 \text{rad}(B) \\ \bar{A} \otimes \text{rad}(B) \end{pmatrix}.$$

By Lemma 3.6, $\widetilde{\text{rad}(B)}$ admits a projective Λ -resolution with all components in $\text{add}(\Lambda e_1)$. This together with the projection from Λe_2 to $\Lambda/\Lambda e_1\Lambda$ provides a projective Λ -resolution of $\Lambda/\Lambda e_1\Lambda$ with the desired property. \checkmark

Corollary 3.7. *The following statements are equivalent:*

- (i) *both A and B have finite global dimension;*
- (ii) *the matrix algebra Λ has finite global dimension;*
- (iii) *the trivially twisted tensor product $A \otimes_0 B$ has finite global dimension.*

Proof. Applying Proposition 2.1 (i) to the stratifying recollements (R1) and (R2) in Proposition 3.5 we obtain the equivalence between (i) and (ii) and between (i) and (iii), respectively. \checkmark

The following theorem is the main result of this paper.

Theorem 3.8. *In the following two cases the matrix algebra Λ does not satisfy the derived Jordan–Hölder property:*

- (1) *both A and B are derived simple but nonsimple;*
- (2) *one of A and B is derived simple but nonsimple, and the other one has infinite global dimension.*

Proof. For a stratification \mathcal{S} of $\mathcal{D}(A)$ let $s(\mathcal{S})$ denote the multiplicity of k as simple factors of \mathcal{S} .

Consider the two stratifying recollements (R1) and (R2) in Proposition 3.5. Recall that $\Lambda/\Lambda e_1\Lambda \cong \bar{A} \otimes \bar{B}$, $e_1\Lambda e_1 = A \otimes_0 B$, $\Lambda/\Lambda e_2\Lambda \cong A \otimes \bar{B}$ and $e_2\Lambda e_2 = \bar{A} \otimes B$. By [4, Lemma 7.1], $s(\mathcal{S}_{\Lambda/\Lambda e_1\Lambda}) = \text{rank}(A)\text{rank}(B)$ for any stratification $\mathcal{S}_{\Lambda/\Lambda e_1\Lambda}$ of $\mathcal{D}(\Lambda/\Lambda e_1\Lambda)$ and hence $s(\mathcal{S}_1) \geq \text{rank}(A)\text{rank}(B)$ for any stratification \mathcal{S}_1 of $\mathcal{D}(A)$ starting from (R1).

We will show that in both cases (1) and (2), $s(\mathcal{S}_2) < \text{rank}(A)\text{rank}(B)$ for any stratification \mathcal{S}_2 starting from the recollement (R2). It follows that DJHP fails for Λ . In both cases we may assume that A is derived simple but nonsimple. Then it follows from [4, Lemma 7.1] that any stratification of $\mathcal{D}(\Lambda/\Lambda e_2\Lambda)$ has simple factors $\text{rank}(B)$ copies of A . Therefore it suffices to show that $s(\mathcal{S}_{e_2\Lambda e_2}) < \text{rank}(A)\text{rank}(B)$ for any stratification $\mathcal{S}_{e_2\Lambda e_2}$ of $\mathcal{D}(e_2\Lambda e_2)$. In view of Lemma 2.3, this is equivalent to showing that $s(\mathcal{S}_B) < \text{rank}(B)$ for any stratification \mathcal{S}_B of $\mathcal{D}(B)$. This is true in case (1) because B is derived simple nonsimple. In case (2) this follows from Proposition 2.1 since B has infinite global dimension. \checkmark

Remark 3.9. The *length* of a stratification is the number of its simple factors. In case (1) of Theorem 3.8, if A and B are not local, namely, the ranks of A and B are greater than 1, then the

length of any stratification starting from $(R1)$ is different from that of any stratification starting from $(R2)$. Indeed, it follows from the proof of Theorem 3.8 that the length of any stratification starting from $(R1)$ is greater than or equal to $1 + \text{rank}(A)\text{rank}(B)$, while the length of any stratification starting from $(R2)$ is precisely $\text{rank}(A) + \text{rank}(B)$.

Remark 3.10. If A (respectively, B) does not satisfy DJHP, nor does Λ . Indeed, we can start with the recollement $(R2)$ and obtain stratifications with different simple factors. Assume that both A and B satisfy DJHP. Following the proof of Theorem 3.8 we see that Λ does not satisfy DJHP if $s(\mathcal{S}_A)\text{rank}(B) + s(\mathcal{S}_B)\text{rank}(A) < \text{rank}(A)\text{rank}(B)$ for a/any stratification \mathcal{S}_A of $\mathcal{D}(A)$ and a/any stratification \mathcal{S}_B of $\mathcal{D}(B)$. For example, this is the case if a/any stratification of $\mathcal{D}(A)$ (respectively, $\mathcal{D}(B)$) has no simple factors k , and B (respectively, A) has infinite global dimension or a/any stratification of $\mathcal{D}(B)$ (respectively, $\mathcal{D}(A)$) has length strictly smaller than $\text{rank}(B)$.

Proposition 2.4 states that over an algebraically closed field all rank 2 algebras of finite global dimension satisfy DJHP. This is far from being true for rank 2 algebras of infinite global dimension.

Corollary 3.11. *Assume that both A and B are local nonsimple. Then Λ does not satisfy DJHP.*

Proof. This follows from Theorem 3.8 because local algebras are derived simple by [2, Proposition 4.11]. ✓

We will see in Example 4.3 that $\Lambda(k[x]/(x^2), k[x]/(x^2))$ is the algebra in [4, Example 7.6] which does not satisfy DJHP.

Examples of derived simple algebras of positive finite global dimension are given in [13, 17, 16]. So the next corollary shows that there are algebras of finite global dimension which do not satisfy DJHP. In [14, Proposition 3.4] Kalck provides a family of finite dimensional algebras of global dimension 2 (hence quasi-hereditary) for which DJHP fails.

Corollary 3.12. *Assume that both A and B are derived simple of positive finite global dimension. Then Λ has finite global dimension and does not satisfy DJHP.*

Proof. By Theorem 3.8, it suffices to show that Λ has finite global dimension, which follows from Corollary 3.7. ✓

The next corollary together with Lemma 3.4 shows that the cardinality of the elementary algebras which do not satisfy DJHP is the same as the cardinality of all elementary algebras.

Corollary 3.13. *Assume that A is local nonsimple. Then $\Lambda(A, \Lambda(A, B))$ does not satisfy DJHP.*

Proof. By Theorem 3.8, it suffices to show that $\Lambda(A, B)$ has infinite global dimension, which follows from Corollary 3.7. ✓

4. QUIVER AND RELATIONS

Let A and B be elementary algebras and let

$$\Lambda = \Lambda(A, B) := \begin{pmatrix} A \otimes_0 B & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes B \end{pmatrix},$$

be defined as in Section 3.2. The aim of this section is to describe the quiver with relations of Λ in terms of those of A and B .

Theorem 4.1. *Let $A = kQ/(\mathcal{R})$ and $B = kP/(\mathcal{S})$, where Q and P are finite quivers, and \mathcal{R} and \mathcal{S} are sets of minimal relations of A and B , respectively. Then as an algebra Λ is isomorphic to $k\Gamma/(\mathcal{I})$, where*

- the set Γ_0 of vertices is $\{i^j, j^i \mid i \in Q_0, j \in P_0\}$, namely, it is two copies of $Q_0 \times P_0$;
- the set Γ_1 consists of three types of arrows:
 - $\alpha^j : s(\alpha)^j \rightarrow t(\alpha)^j$, for $\alpha \in Q_1$ and $j \in P_0$,
 - $c(i, j) : i^j \rightarrow j^i$, for $i \in Q_0$ and $j \in P_0$,
 - $\beta^i : s(\beta)^i \rightarrow t(\beta)^i$, for $\beta \in P_1$ and $i \in Q_0$;
- the set \mathcal{I} consists of three types of relations:
 - r^j , for $r \in \mathcal{R}$ and $j \in P_0$,
 - $c(i, j)\alpha^j$, for $i \in Q_0, j \in P_0$ and $\alpha \in Q_1$ with $t(\alpha) = i$,
 - s^i , for $s \in \mathcal{S}$ and $i \in Q_0$,

where

$$r^j = \sum \lambda_{i_1, \dots, i_s} \alpha_{i_1}^j \cdots \alpha_{i_s}^j$$

for $r = \sum \lambda_{i_1, \dots, i_s} \alpha_{i_1} \cdots \alpha_{i_s} \in \mathcal{R}$ and $j \in P_0$, and

$$\begin{aligned} s^i &= \sum \mu_{j_1, \dots, j_t} \beta_{j_1}^i c(i, s(\beta_{j_1})) \beta_{j_2}^i c(i, s(\beta_{j_2})) \cdots \beta_{j_{t-1}}^i c(i, s(\beta_{j_{t-1}})) \beta_{j_t}^i \\ &= \sum \mu_{j_1, \dots, j_t} \beta_{j_1}^i c \beta_{j_2}^i \cdots \beta_{j_{t-1}}^i c \beta_{j_t}^i \end{aligned}$$

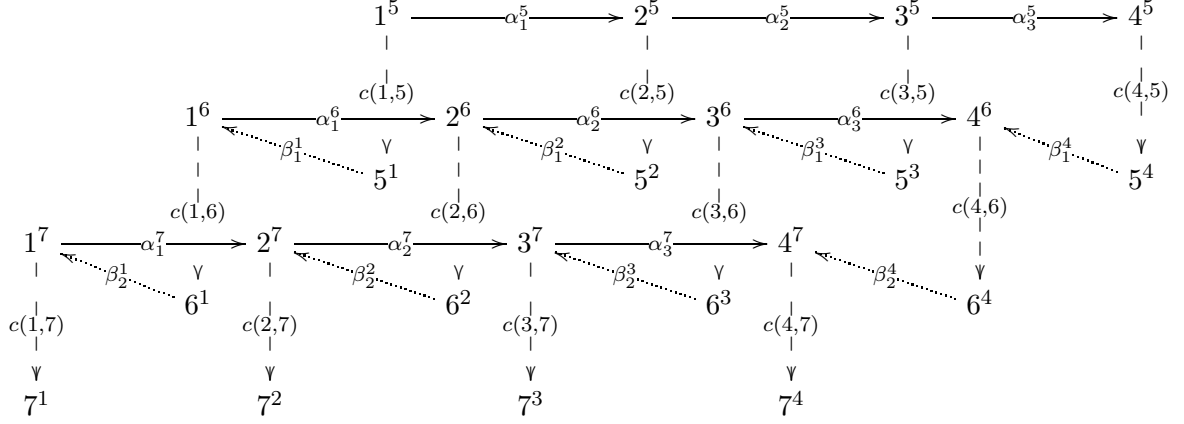
for $s = \sum \mu_{j_1, \dots, j_t} \beta_{j_1} \cdots \beta_{j_t} \in \mathcal{S}$ and $i \in Q_0$, where $c = \sum_{i \in Q_0, j \in P_0} c(i, j)$.

Before proving Theorem 4.1, we first give two examples illustrating the construction of the quiver with relations in the theorem.

Example 4.2. Let Q and P be the following quivers with relations

$$\begin{array}{ccccccc} 1 & \xrightarrow{\alpha_1} & 2 & \xrightarrow{\alpha_2} & 3 & \xrightarrow{\alpha_3} & 4, \alpha_3 \alpha_2 \alpha_1 \\ & & & & 5 & \xrightarrow{\beta_1} & 6 \xrightarrow{\beta_2} 7, \beta_2 \beta_1. \end{array}$$

Then Γ is given by



and \mathcal{I} consists of

- $\alpha_3^j \alpha_2^j \alpha_1^j$, $j = 5, 6, 7$;
- $c(i, j) \alpha_{i-1}^j$, $i = 2, 3, 4$, $j = 5, 6, 7$;
- $\beta_2^i c(i, 6) \beta_1^i$, $i = 1, 2, 3, 4$.

Example 4.3. Let $A = k[x]/(x^2)$ and $B = k[y]/(y^2)$. The quiver of A (respectively, B) has a unique vertex, denoted by a (respectively, b), and one loop α (respectively, β). By Theorem 4.1, the quiver with relations of Λ is

$$\alpha^b \circ a^b \xrightleftharpoons[\beta^a]{c(a,b)} b^a, \quad \beta^a c(a, b) \beta^a, (\alpha^b)^2, c(a, b) \alpha^b.$$

As A and B are both local nonsimple, DJHP fails for Λ by Corollary 3.11. Note that Λ is of rank 2, so a stratification of $\mathcal{D}(\Lambda)$ is just a recollement. The factors of $(R1)$ are $\bar{A} \otimes \bar{B} = k$ and $A \otimes_0 B \cong k\langle x, y \rangle / (x^2, y^2, yx)$, while the factors of $(R2)$ are $A \otimes \bar{B} = k[x]/x^2$ and $\bar{A} \otimes B = k[y]/y^2$, as also shown in [4, Example 7.6].

The rest of this section is devoted to the proof of Theorem 4.1.

Proof of Theorem 4.1. Recall from the proof of Lemma 3.3 (iii) that

$$\begin{aligned} \text{rad}(\Lambda) &= \begin{pmatrix} A \otimes \text{rad}(B) + \text{rad}(A) \otimes B & A \otimes \text{rad}(B) \\ \bar{A} \otimes B & \bar{A} \otimes \text{rad}(B) \end{pmatrix} \\ &= \begin{pmatrix} (\text{rad } A \otimes \bar{B}) \oplus (\bar{A} \otimes \text{rad } B) \oplus (\text{rad } A \otimes_0 \text{rad } B) & A \otimes \text{rad } B \\ \bar{A} \otimes B & \bar{A} \otimes \text{rad } B \end{pmatrix} \end{aligned}$$

and

$$\bar{\Lambda} = \begin{pmatrix} \bar{A} \otimes \bar{B} & 0 \\ 0 & \bar{A} \otimes \bar{B} \end{pmatrix}.$$

We write e_i ($i \in Q_0$) and f_j ($j \in P_0$) for the primitive idempotents of A and B respectively, and e_{ij}, e_{ji} the primitive idempotents in $k\Gamma$ corresponding to i^j and j^i . Then the map

$$e_{ij} \mapsto \begin{pmatrix} e_i \otimes f_j & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{ji} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & e_i \otimes f_j \end{pmatrix}, \quad (i \in Q_0, j \in P_0)$$

defines an algebra isomorphism from $k\Gamma_0$ to $\bar{\Lambda}$. We identify these two semisimple algebras via this isomorphism.

The square of the radical of Λ is

$$\begin{aligned} \text{rad}^2 \Lambda &= (\text{rad} \Lambda)^2 \\ &= \begin{pmatrix} \text{rad}^2(A \otimes_0 B) \oplus (A \otimes \text{rad} B)(\bar{A} \otimes B) & \text{rad}(A \otimes_0 B)A \otimes \text{rad} B + (A \otimes \text{rad} B)(\bar{A} \otimes \text{rad} B) \\ (\bar{A} \otimes B) \text{rad}(A \otimes_0 B) + (\bar{A} \otimes \text{rad} B)(\bar{A} \otimes B) & (\bar{A} \otimes B)(A \otimes \text{rad} B) + \text{rad}^2(\bar{A} \otimes B) \end{pmatrix} \\ &= \begin{pmatrix} (\text{rad}^2 A \otimes \bar{B}) \oplus (A \otimes \text{rad} B) & (\text{rad} A \otimes \text{rad} B) \oplus (\bar{A} \otimes \text{rad}^2 B) \\ \bar{A} \otimes \text{rad} B & \bar{A} \otimes \text{rad} B \end{pmatrix}. \end{aligned}$$

Hence

$$\text{rad} \Lambda / \text{rad}^2 \Lambda = \begin{pmatrix} (\text{rad} A / \text{rad}^2 A) \otimes \bar{B} & \bar{A} \otimes (\text{rad} B / \text{rad}^2 B) \\ \bar{A} \otimes \bar{B} & 0 \end{pmatrix}.$$

It follows that the map

$$\alpha^j \mapsto \begin{pmatrix} \alpha \otimes f_j & 0 \\ 0 & 0 \end{pmatrix}, \quad c(i, j) \mapsto \begin{pmatrix} 0 & 0 \\ e_i \otimes f_j & 0 \end{pmatrix}, \quad \beta^i \mapsto \begin{pmatrix} 0 & e_i \otimes \beta \\ 0 & 0 \end{pmatrix} \quad (i \in Q_0, j \in P_0)$$

defines a $\bar{\Lambda}$ - $\bar{\Lambda}$ -bimodule isomorphism from the k -vector space spanned by the arrows of Γ to $\text{rad}(\Lambda) / \text{rad}^2(\Lambda)$. This isomorphism then extends to a surjective algebra homomorphism

$$\tilde{\Phi} : k\Gamma \twoheadrightarrow \Lambda.$$

For any path $q = \alpha_1 \alpha_2 \cdots \alpha_s$ of Q and $j \in P_0$, $q^j := \alpha_1^j \alpha_2^j \cdots \alpha_s^j$ is a path of Γ and we have

$$\tilde{\Phi}(q^j) = \begin{pmatrix} q \otimes f_j & 0 \\ 0 & 0 \end{pmatrix}.$$

For any path $p = \beta_1 \beta_2 \cdots \beta_s$ of P and $i \in Q_0$ we have

$$\begin{aligned} \tilde{\Phi}(c\beta_1^i c\beta_2^i c \cdots c\beta_s^i c) &= \begin{pmatrix} 0 & 0 \\ e_i \otimes \beta & 0 \end{pmatrix}, \\ \tilde{\Phi}(c\beta_1^i c\beta_2^i c \cdots c\beta_s^i) &= \begin{pmatrix} 0 & 0 \\ 0 & e_i \otimes \beta \end{pmatrix}, \\ \tilde{\Phi}(\beta_1^i c\beta_2^i c \cdots c\beta_s^i) &= \begin{pmatrix} 0 & e_i \otimes \beta \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, the three types of relations in \mathcal{I} belong to the kernel of $\tilde{\Phi}$: for $r \in \mathcal{R}$ and $j \in P_0$,

$$\tilde{\Phi}(r^j) = \begin{pmatrix} r \otimes f_j & 0 \\ 0 & 0 \end{pmatrix} = 0;$$

for $i \in Q_0$, $j \in P_0$ and $\alpha \in Q_1$ with $t(a) = i$,

$$\tilde{\Phi}(c(i, j)\alpha^j) = \begin{pmatrix} 0 & 0 \\ e_i \otimes f_j & 0 \end{pmatrix} \begin{pmatrix} \alpha \otimes f_j & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha \otimes f_j & 0 \end{pmatrix} = 0;$$

for $s = \sum \mu_{j_1, \dots, j_t} \beta_{j_1} \cdots \beta_{j_t} \in \mathcal{S}$ and $i \in Q_0$,

$$\begin{aligned} \tilde{\Phi}(s^i) &= \tilde{\Phi}\left(\sum \mu_{j_1, \dots, j_t} \beta_{j_1}^i c \beta_{j_2}^i c \cdots \beta_{j_{t-1}}^i c \beta_{j_t}^i\right) \\ &= \sum \mu_{j_1, \dots, j_t} \begin{pmatrix} 0 & e_i \otimes \beta_{j_1} \beta_{j_2} \cdots \beta_{j_t} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & e_i \otimes s \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

Hence $\tilde{\Phi}$ factors through a surjective algebra homomorphism

$$\Phi : \tilde{\Lambda} = k\Gamma/(\mathcal{I}) \twoheadrightarrow \Lambda.$$

We will show that Φ is bijective and thus an algebra isomorphism.

By abuse of notation we denote by e_1 the sum of all the idempotents e_{ij} 's and e_2 the sum of all e_{ji} 's in $\tilde{\Lambda}$. Then $\Phi(e_1) = e_1$ and $\Phi(e_2) = e_2$. Since e_1 and e_2 are orthogonal to each other, both algebras $\tilde{\Lambda}$ and Λ decompose as 2×2 -matrix algebras and the decomposition is compatible with Φ . We write Φ_{ij} for the induced map from $e_i \tilde{\Lambda} e_j$ to $e_i \Lambda e_j$ ($i, j = 1, 2$). Then $\Phi = \bigoplus_{i,j=1}^2 \Phi_{ij}$. The bijectivity of Φ is equivalent to the bijectivity of all Φ_{ij} .

Bijectivity of $\Phi_{11} : e_1 \tilde{\Lambda} e_1 \rightarrow e_1 \Lambda e_1 = A \otimes_0 B$: The trivially twisted tensor product $A \otimes_0 B$ as a k -algebra is generated by $e_i \otimes f_j$, $\alpha \otimes f_j$ and $e_i \otimes \beta$ for $i \in Q_0$, $j \in P_0$, $\alpha \in Q_1$ and $\beta \in P_1$, with generating relations $r \otimes f_j$ for $r \in \mathcal{R}$, $e_i \otimes s$ for $s \in \mathcal{S}$ and $(e_i \otimes \beta)(\alpha \otimes f_j)$. The preimages of $\alpha \otimes f_j$ and $e_i \otimes \beta$ in $e_1 \tilde{\Lambda} e_1$ are α^j and $\beta^i c$ respectively. They also satisfy the generating relations of $A \otimes_0 B$. Indeed for $r = \sum \lambda_{i_1, \dots, i_s} \alpha_{i_1} \cdots \alpha_{i_s} \in \mathcal{R}$, $r^j = \sum \lambda_{i_1, \dots, i_s} \alpha_{i_1}^j \cdots \alpha_{i_s}^j = 0$. For $s = \sum \mu_{j_1, \dots, j_t} \beta_{j_1} \cdots \beta_{j_t} \in \mathcal{S}$, $\sum \mu_{j_1, \dots, j_t} \beta_{j_1}^i c \beta_{j_2}^i c \cdots \beta_{j_t}^i c = s^i c = 0$. For $i \in Q_0$, $j \in P_0$, $\alpha \in Q_1$ and $\beta \in P_1$, we have $\beta^i c \alpha^j = 0$. Note that $e_1 \tilde{\Lambda} e_1$ as a k -algebra is generated by e_{ij} , α^j and $\beta^i c$. Therefore Φ_{11} must be bijective.

Bijectivity of $\Phi_{22} : e_2 \tilde{\Lambda} e_2 \rightarrow e_2 \Lambda e_2 = \bar{A} \otimes B$. The tensor product $\bar{A} \otimes B$ as a k -algebra is generated by $e_i \otimes f_j$ and $e_i \otimes \beta$ for $i \in Q_0$ and $\beta \in P_1$ with generating relations $e_i \otimes s$ for $s \in \mathcal{S}$. The preimages of $e_i \otimes f_j$ and $e_i \otimes \beta$ in $e_2 \tilde{\Lambda} e_2$ are e_{ji} and $c \beta^i$ respectively. They satisfy the generating relations of $\bar{A} \otimes B$. Indeed for $s = \sum \mu_{j_1, \dots, j_t} \beta_{j_1} \cdots \beta_{j_t} \in \mathcal{S}$, $c s^i = \sum \mu_{j_1, \dots, j_t} c \beta_{j_1}^i c \beta_{j_2}^i c \cdots \beta_{j_t}^i = 0$. An a k -algebra $e_2 \tilde{\Lambda} e_2$ is generated by e_{ji} and $c \beta^i$, so Φ_{12} must be bijective.

Bijectivity of $\Phi_{21} : e_2 \tilde{\Lambda} e_1 \rightarrow e_2 \Lambda e_1 = \bar{A} \otimes B$. For $i \in Q_0$, $j \in P_0$ and $\beta \in P_1$, the preimages of $e_i \otimes f_j$ and $e_i \otimes \beta$ are $c(i, j)$ and $c \beta^i c$ respectively. Consider the right multiplication map r_c from $e_2 \tilde{\Lambda} e_2$ to $e_2 \tilde{\Lambda} e_1$, sending an element x to xc . This map is surjective according to the shape of Γ and the description of \mathcal{I} . Hence $\dim_k e_2 \tilde{\Lambda} e_1 \leq \dim_k e_2 \tilde{\Lambda} e_2 = \dim_k e_2 \Lambda e_2 = \dim_k e_2 \Lambda e_1$. But the surjectivity of Φ_{21} implies $\dim_k e_2 \tilde{\Lambda} e_1 \geq \dim_k e_2 \Lambda e_1$. Hence $e_2 \tilde{\Lambda} e_1$ and $e_2 \Lambda e_1$ have the same dimension and Φ_{21} is bijective.

Finally for the bijectivity of $\Phi_{12} : e_1 \tilde{\Lambda} e_2 \rightarrow e_1 \Lambda e_2 = A \otimes \text{rad } B$, consider for each $c(i, j)$ the right multiplication map $r_{c(i, j)} : \tilde{\Lambda} e_{ji} \rightarrow \tilde{\Lambda} e_{ij}$, and the sum $r_c = \sum r_{c(i, j)} : \tilde{\Lambda} e_2 \rightarrow \tilde{\Lambda} e_1$. By Lemma 4.4 below, each $r_{c(i, j)}$ is injective, so is r_c . Let $r_c^1 : e_1 \tilde{\Lambda} e_2 \rightarrow e_1 \tilde{\Lambda} e_1$ be the corresponding

summand of r_c . It follows that r_c^1 is injective. The following commutative diagram

$$\begin{array}{ccc} e_1 \tilde{\Lambda} e_2 & \xrightarrow{r_c^1} & e_1 \tilde{\Lambda} e_1 \\ \Phi_{12} \downarrow & & \downarrow \Phi_{11} \\ e_1 \Lambda e_2 = A \otimes \text{rad } B & \xrightarrow{\text{inc}} & A \otimes_0 B = e_1 \Lambda e_1 \end{array}$$

shows that Φ_{12} is injective, and hence bijective. \checkmark

Lemma 4.4. *Let $A = kQ/(\mathcal{R})$ where Q is a finite quiver and \mathcal{R} is a set of minimal relations. Suppose $c : s(c) \rightarrow t(c)$ is a path in Q such that any path appearing in \mathcal{R} is not of the form pc for some path p in Q . Then the right multiplication map $r_c : Ae_{t(c)} \rightarrow Ae_{s(c)}$, $x \mapsto xc$, is injective.*

Proof. Put $\tilde{A} = kQ$ and $I = (\mathcal{R})$. We have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ie_{t(c)} & \longrightarrow & \tilde{A}e_{t(c)} & \longrightarrow & Ae_{t(c)} \longrightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & r_c \downarrow \\ 0 & \longrightarrow & Ie_{s(c)} & \longrightarrow & \tilde{A}e_{s(c)} & \longrightarrow & Ae_{s(c)} \longrightarrow 0 \end{array}$$

where the rows are exact and all the vertical maps are right multiplication with c . It is clear that α and β are injective. Hence r_c is injective if and only if $\text{Im}(\beta) \cap Ie_{s(c)} = \text{Im}(\alpha)$, i.e. $\tilde{A}c \cap Ie_{s(c)} = Ic$ in \tilde{A} . It suffices to show ' \subseteq ', i.e. if $x \in \tilde{A}$ satisfies $xc \in Ie_{s(c)}$, then $x \in I$. Write $xc = \sum_{r \in \mathcal{R}} a_1 r a_2$ with $a_1, a_2 \in \tilde{A}$, and write further $a_2 = a'_2 + a''_2$ such that all paths appearing in a'_2 are of the form pc for some path p in \tilde{A} , and none of the paths appearing in a''_2 is of such form. Then $xc = \sum_{r \in \mathcal{R}} a_1 r a'_2 + \sum_{r \in \mathcal{R}} a_1 r a''_2$ in \tilde{A} . The first summand is of the form $x'c$ for some $x' \in I$ and the second summand must be zero. Hence $xc = x'c$, namely $\beta(x) = \beta(x')$. It follows from the injectivity of β that $x = x' \in I$. \checkmark

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